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AN INCREMENTAL APPROACH TO THE SOLUTION OF SNAPPING AND BUCKLING PROBLEMS†

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Abstract—This paper is concerned with the numerical solution of systems of equations of discrete variables, which represent the nonlinear behaviour of elastic systems under conservative loading conditions. In particular, an incremental approach to the solution of buckling and snapping problems is explored.

The topics that are covered can be summarized as follows:—The computation of nonlinear equilibrium paths with continuation through limit points and bifurcation points.—The determination of critical equilibrium states.

Characteristic to the procedures employed is the use of the length of the equilibrium path as control parameter. This feature, together with the second order iteration method of Newton, offers a reliable basis for the procedures described. Actual computations, carried out on a finite element model of a shallow circular arch, illustrate the effectiveness of the methods proposed.

1. INTRODUCTION

The elastic stability analysis of plate and shell structures by means of the finite element method or finite difference method inevitably involves the solution of large systems of nonlinear equations. In principle, two broad classes of solution methods are available today. One consists of computer adaptations of a perturbation method developed as far back as 1945, by Koiter[1, 2]. Examples of this approach in terms of suggested, or actual programme implementations are given by[5–9], in particular[7] and [8]. The other approach, in existence since the evolution of the present day computer, covers methods which try to solve the equations step by step, i.e. in a pointwise fashion. Examples of this mode of attack can be found in[10–22]. The present paper is exclusively devoted to this second type of approach.

It is a consequence of any discrete formulation (e.g. the finite element method) that the deformation of a given structure is described by a set of (N) deformation parameters, also called generalized coordinates. In this context the load-deformation history of a structure presents itself as a curve in a ($N+1$) dimensional space spanned by the deformation parameters and the magnitude of the applied loads. Such a curve is usually referred to as equilibrium path or deformation path. The problem of elastic stability is intimately connected with singularities that occur somewhere along the path under consideration (usually the path which is connected with the undeformed state of the structure, the so-called primary path). These singular points are better known as critical points. Well known is their classification into limit points and bifurcation points, geometrical concepts that are connected with the physical concepts of snapping and buckling respectively[3]. In principle, the stability analysis of a structure or structural system consists of a computation and evaluation of the critical point of the path that is considered to be relevant to the problem at hand. To be able to carry out such analysis, a computational procedure should have the capability: (1) to compute the critical points, i.e. limit or bifurcation points, (2) to trace parts of the path or path's (branches) connected with these points‡.

The numerical strategies set forward in this paper are designed to meet these requirements. They are based on an incremental method proposed earlier in[14, 15] which features Newton's method and a special parameter controlling the progress of the computations along the equilibrium path(s). In the geometrical terms the control parameter selected corresponds, in good approximation to the "arc length" of the equilibrium path to be computed. It is introduced by means of an auxiliary equation which is added to the set of equations governing the

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‡This is another way of saying that the method should have the capability to compute post-critical equilibrium states.

equilibrium of the structure at hand. An important advantage of this particular choice is that the modified set of equations become singular only at a bifurcation point. This means that the domain of application of the method is increased considerably in comparison with that of more conventional methods.

The present paper is a shortened version of [16]† where the application of this method to the problems of the type (1) + (2) was studied in detail. Just as in [16], the emphasis will here be put on an exposition of possible strategies and an evaluation of their feasibility.

The reliability of Newton's method in procedures of the type considered presently is well known [23], and it is this property which prompted its choice. A serious drawback is however, that it requires a large volume of operations when the dimension of the system of equations is large. This detracts, at first sight, much from the practical significance of this method in connection with the solution of large scale problems. Nevertheless, the use of Newton's method makes it possible to deal with the question of effectiveness first which is the main objective of this paper. It should be realised however, that improvements in the economy are possible by measures which do not impair the intrinsic advantages of the methods presently proposed. Some of these modifications are mentioned, but a directed discussion of possible ways to improve the computational efficiency will be deferred to a future paper.

The references [10–13, 17–22] constitute a sample of contributions to the present subject, which were readily available at the time [16] was completed. The principal difference between the present paper and the references mentioned is the difference in choice of the control parameter in the computations. In [10, 11] a family of control parameters is considered, which are suitable for the continuation of the computations through limit points. However, the selection of the one to be applied is left to depend on the particular problem at hand. In the examples presented, the parameter chosen is called for only in a close neighbourhood of the limit point. In the other references, the difficulties connected with limit points are either circumvented by a "jump" technique [22], or solved by replacing the load by one of the displacement variables as control parameter [12, 13, 21, 22]. Predictor-corrector schemes are applied with either Newton's method as corrector [10–13, 17, 18] or a modification of this method: the so-called direct substitution method or modified Newton–Raphson method [19–22, 29]. The question of continuation onto the branch of a bifurcation point is discussed in [13, 22].

The subject of the present paper will be preceded with a summary of some elementary concepts of elastic stability theory for as far as this appears to be necessary for the developments that follow.

BASIC CONCEPTS

2.1 *Equilibrium equations*

The class of problems considered here concerns purely elastic structures under conservative loading conditions. It is well known that for these structures a potential energy function can be defined from which all pertinent information as regards their behaviour can be derived. It is assumed here that the kinematical configurations of a structure are described by a finite set of generalized coordinates:

$$\tilde{t} = [t_1, t_2, t_3, \ldots, t_N] \tag{2.1}$$

which are also referred to as displacement variables or deformation parameters. As is customary in the theory of elastic stability, the loading of the structure is supposed to be dependent on a single intensity parameter, here denoted by ρ.

Let the potential energy be given by the function

$$P = P(\tilde{t}; \rho). \tag{2.2}$$

The equilibrium of the structure is then determined by a stationary value of this function. Stationary values of the energy P are governed by the set of equations

$$f_i(\tilde{t}; \rho) = \frac{\partial P(\tilde{t}; \rho)}{\partial t_i} = 0, \quad i = 1, 2, \ldots N. \tag{2.3}$$

†This report was distributed only on a limited scale.

In general, these equations are nonlinear in $\tilde{t}$ and ρ, and this is always assumed to be the case in the sequel. The co-ordinates t_i are supposed to be defined in such a way that the undeformed state is the reference state

$$\rho = 0; \quad \tilde{t} = \tilde{0}. \tag{2.4}$$

The configuration $[\rho; \tilde{t}]$ of the structure can be visualized as a point in a $(N+1)$ dimensional Euclidean Space R_{N+1}, with an orthonormal basis

$$[\tilde{e}_i] = [\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \ldots \tilde{e}_N]. \tag{2.5}$$

This point is then given by

$$[\rho, \tilde{t}] = \rho\tilde{e}_0 + t_i\tilde{e}_i. \tag{2.6}$$

Here and in the following repeated indices imply the well known summation convention.

In general, more than one solution $\tilde{t}$ exists at a given value of the load ρ. The solutions vary when the load ρ is varied and describe curves in R_{N+1}, the equilibrium paths of the structure. They can be described in parametric form by

$$\rho = \rho(\eta); \tilde{t} = \tilde{t}(\eta) \tag{2.7}$$

where η is a suitably chosen path parameter. In the physical model of the structures considered, the load is prescribed, which implies that ρ takes the role of path parameter

$$\rho = \rho; \tilde{t} = \tilde{t}(\rho). \tag{2.8}$$

For the mathematical model the choice (2.8) is not always appropriate. Here, it will prove to be more useful to choose η to satisfy

$$\left(\frac{d\rho}{d\eta}\right)^2 + \frac{dt_k}{d\eta}\frac{dt_k}{d\eta} = 1. \tag{2.9}$$

This defines η as the arc length of the curve (2.7) and it will be denoted by "s" in the sequel.

2.2 *Critical points*

According to the theory of stability of conservative systems, the stability of the equilibrium configurations (2.8) is ensured if the quadratic form

$$P_2(\Delta\tilde{t}) = \frac{1}{2}\frac{\partial^2 P(\tilde{t}; \rho)}{\partial t_i \partial t_j}\Delta t_i \Delta t_j = \frac{1}{2} f_{ij}\Delta t_i \Delta t_j \tag{2.10}$$

is positive definite. They are unstable if $P_2(\Delta\tilde{t})$ is indefinite, while the transition between a stable and unstable point of a path is marked by a positive semi-definite $P_2(\Delta\tilde{t})$. This point of the path is called critical state and it will usually be denoted by an asterisk.

$$[\rho^*, \tilde{t}^*] = [\rho(s^*), \tilde{t}(s^*)]. \tag{2.11}$$

The coefficient matrix of the quadratic form $P_2(\Delta\tilde{t})$ is symmetric. Using the notation for partial differentiation introduced in (2.10), this important property is expressed by

$$f_{ij} = f_{ji}. \tag{2.12}$$

The precise definition of a critical equilibrium state is given by means of the generalized eigenvalue problem

$$\{f_{ij}(\tilde{t}; \rho) - \omega(k)T_{ij}\}a_j(k) = 0. \tag{2.13}$$

Here, $\tilde{a}(k)$ denote the characteristic vectors and $\omega(k)$ the characteristic values of $[f_{i,j}]$ at a given configuration $[\rho, \tilde{t}]$ with respect to $[T_{i,j}]$: a positive definite, but otherwise arbitrary matrix†. This freedom in the choice of $[T_{i,j}]$ is only of minor importance in the present paper. With the exception of Section 5.1 eqns (5.2), $[T_{i,j}]$ will here always be equated to the identity matrix, thus

$$T_{i,j} = \delta_{ij} (= \text{the Kronecker delta}). \tag{2.14}$$

The characteristic values $\omega(k)$ are supposed to be arranged according to the sequence

$$\omega(1) \leqq \omega(2) \leqq \omega(3) \leqq \cdots \leqq \omega(N). \tag{2.15}$$

The quadratic form $P_2(\Delta\tilde{t})$ is positive definite, and therefore the equilibrium configuration $[\rho(s); \tilde{t}(s)]$ is stable as long as the characteristic values $\omega(k)$ satisfy

$$0 < \omega(1) \leqq \omega(2) \leqq \cdots \leqq \omega(N). \tag{2.16}$$

The stability limit is reached when

$$0 = \omega(1) = \omega(2) = \cdots = \omega(K) < \omega(K+1) \leqq \cdots \leqq \omega(N) \tag{2.17}$$

thus when one or more eigenvalues $\omega(k)$ in the sequence (2.15) are zero. In this paper the restriction will be made to discrete critical points only, i.e. to points defined by

$$0 = \omega(1) < \omega(2) \leqq \omega(3) \leqq \cdots \leqq \omega(N). \tag{2.18}$$

In the context of the theory of elastic stability of conservative systems, (2.17) and (2.18) are unambigous definitions of a critical state. There is another criterion which is useful in applications but which can only be appreciated as a necessary condition. The determinant

$$D = \det\{f_{i,j}(\tilde{t}; \rho)\} \tag{2.19}$$

as is wellknown, can be computed from

$$D = \left\{\prod_{k=1}^{N} \omega(k)\right\} \det\{T_{i,j}\}. \tag{2.20}$$

It follows that at a critical state

$$D = D(\tilde{t}^*; \rho^*) = 0. \tag{2.21}$$

The criteria (2.18) and (2.21) will prove useful instruments for the determination of critical points.

2.3 *Limit point and bifurcation point*

It is well known that in general, equilibrium becomes critical in either a limit point or bifurcation point (Fig. 1). These geometrical concepts follow from the analysis of the behaviour of the solutions (2.8) in the neighbourhood of the critical points (see, e.g. [3, 4]). A complete analysis of this nature is extensive and falls outside the scope of this paper. In what follows it is only necessary to know the criteria which characterize the difference and to have at the disposal a means to compute the direction vector of the branch of a bifurcation point, whenever such point is encountered. The present and the following section will deal with these questions.

†It is convenient to consider the vectors $\tilde{a}(k)$ normalised according to $|\tilde{a}(k)| = 1$, where $|\ |$ denotes the Euclidean norm: $|\tilde{a}| = (\tilde{a}\tilde{a})^{1/2} = (a_i a_i)^{1/2}$. Note that the inner product will be denoted by either $(\tilde{a}\tilde{b})$ or $(a_i b_i)$.

The properties of the path are described by the path derivatives

$$[\dot{\rho}, \dot{\tilde{t}}] = \left[\frac{d\rho}{ds}, \frac{d\tilde{t}}{ds}\right]$$
$$[\ddot{\rho}, \ddot{\tilde{t}}] = \left[\frac{d^2\rho}{ds^2}, \frac{d^2\tilde{t}}{ss^2}\right], \quad \text{etc.} \tag{2.22}$$

These vector quantities are determined by the following equations

$$\frac{d}{ds}[f_i(\tilde{t}; \rho)] = f_{i,j}\dot{t}_j + f_{i,0}\dot{\rho} = 0 \tag{2.23a}$$

$$\frac{d^2}{ds^2}[f_i(\tilde{t}; \rho)] = f_{i,j}\ddot{t}_j + f_{i,0}\ddot{\rho} + (\dot{f}_{i,j}\dot{t}_j + \dot{f}_{i,0}\dot{\rho}) = 0, \text{ etc.} \tag{2.23b}$$

Equations (2.23) are not complete without

$$\dot{\rho}\dot{\rho} + \dot{t}_h\dot{t}_h = 1, \tag{2.24a}$$
$$\dot{\rho}\ddot{\rho} + \dot{t}_h\ddot{t}_h = 0, \text{ etc.} \tag{2.24b}$$

which side conditions follow from the definition of the path parameter s, see eqn (2.9). It is noted that in (2.23) the convention $f_{i,0} = \partial f_i/\partial\rho$ is introduced. Equations (2.23), in conjunction with (2.24) should hold everywhere along the path $[\rho(s), \tilde{t}(s)]$. However, at the critical point, $[f_{i,j}]$ is singular in accordance to definition (2.21). This means that the non-homogeneous parts of eqns (2.23) should satisfy the orthogonality conditions

$$[a_i(1)f_{i,0}\dot{\rho}]_{s=s^*} = 0 \tag{2.25a}$$
$$[a_i(1)\{f_{i,0}\ddot{\rho} + \dot{f}_{i,j}\dot{t}_j + \dot{f}_{i,0}\dot{\rho}\}]_{s=s^*} = 0 \tag{2.25b}$$

where the asterisk indicates that the evaluation is made at a critical state†.

It follows from the first requirement (2.25a), that two cases should be considered seperately

$$\dot{\rho}(s^*) = 0 \tag{2.26a}$$
$$f_{i,0}(s^*)a_i(1) = 0. \tag{2.26b}$$

In the general theory, these conditions are known as the condition for a limit point and bifurcation point respectively. Of course, they should be read in conjunction with (2.18) or (2.21).

An equivalent formulation of (2.26) can be presented which does not contain the eigenvector $\tilde{a}(1)$. It is first noted that eqns (2.23a) can be rewritten in the form

$$D\dot{t}_i = d_i\dot{\rho} \tag{2.27}$$

where the components d_i of the vector $\tilde{d}$ are defined by

$$d_i = -\frac{\partial D}{\partial f_{j,i}} f_{j,0} \qquad [14\text{–}16]. \tag{2.28}$$

The transformation of (2.23a)–(2.27) is accomplished by means of the matrix

$$[A_{ij}] = \left[\frac{\partial D}{\partial f_{j,i}}\right] \tag{2.29}$$

†Equations (2.25) are sometimes referred to as consistency relations.

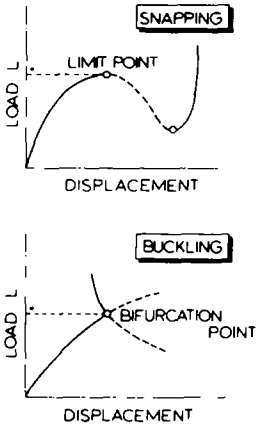

Fig. 1. Problem types.

which is the matrix of cofactors of D. Note the relationship

$$\frac{\partial D}{\partial f_{j,i}} \cdot f_{j,k} = D\delta_{ik}. \tag{2.30}$$

As before, examination of the transformed eqns (2.27) subject to (2.18), leads to the conclusion that two different situations should be considered.

$$\dot{\rho}(s^*) = 0 \tag{2.31a}$$

$$d_i(s^*) = -\frac{\partial D}{\partial f_{h,i}} f_{h,0} = 0. \tag{2.31b}$$

In the form (2.31b) the bifurcation condition is less known than (2.26b). However, it will prove to be a useful relation in what follows. A proof of the equivalence of (2.31)–(2.26) is given in the Appendix.

2.4 *The tangent to the branch*

As was mentioned earlier, the character of a critical point can be exposed by studying the solutions of (2.23) at such a point. At present it will suffice to restrict the considerations to the solution for the tangent $[\dot{\rho}(s^*), \dot{\bar{t}}(s^*)]$. Only a sketch of the derivations will be given.

The computation of the tangent at the critical point is carried out on the basis of the observation that a solution $\dot{\bar{t}}$ of (2.23a) can be written as the sum of the particular solution $\dot{\rho}\bar{y}$ and an arbitrary multiple of the eigenvector $\bar{a}(1)$ associated with the critical state, thus

$$\dot{\bar{t}} = \dot{\rho}\bar{y} + \mu\bar{a}(1). \tag{2.32}$$

It is noted in passing, that according to this proposition

$$f_{i,j}(s^*)y_j + f_{i,0}(s^*) = 0; \quad f_{i,j}(s^*)a_j(1) = 0; \quad y_i a_i(1) = 0. \tag{2.33}$$

When expression (2.32) is subjected to conditions (2.26a) and (2.24a), it follows that

$$[\dot{\rho}(s^*), \dot{\bar{t}}(s^*)] = [0, \bar{a}(1)]. \tag{2.34}$$

Consequently, at the critical point defined by $\dot{\rho}(s^*) = 0$, the tangent vector corresponds to the eigenvector $\bar{a}(1)$.

In the second case (2.26b), the situation is more complex. The general solution (2.32) is now not determined by (2.25a) alone, but also needs the side condition (2.25b) for its completion.

The result is that two solutions emerge

$$[\dot{\rho}(s^*), \dot{\bar{t}}(s^*)_I] = [\dot{\rho}_I, \dot{\rho}_I\bar{y} + \mu_I\bar{a}]$$
$$[\dot{\rho}(s^*), \dot{\bar{t}}(s^*)_{II}] = [\dot{\rho}_{II}, \dot{\rho}_{II}\bar{y} + \mu_{II}\bar{a}]\dagger \tag{2.35}$$

where, for convenience $\bar{a}$ is written for $\bar{a}$ (1). The specification of $\dot{\rho}_I$, $\dot{\rho}_{II}$, μ_I, μ_{II}, $\bar{y}$ will not be given because the explicit form of the solution (2.35) is not of interest here. This follows because once the bifurcation point is computed, one of the solutions (2.35) will automatically be provided by the methods to be discussed in this paper. It is advantageous therefore, to use this tangent vector in the construction of the other. Note, that it is possible to express one of the two solutions (2.35) as a linear combination of the other and the eigenmode $\bar{a}$. A convenient choice of such construction is

$$[\dot{\rho}_{II}, \dot{\bar{t}}_{II}] = \alpha[\mu\dot{\rho}_I, \mu\dot{\bar{t}}_I + \bar{a}]. \tag{2.36}$$

If $[\dot{\rho}_I, \dot{\bar{t}}]$ is known explicitly together with the eigenmode $\bar{a}$, α and μ can be computed from the requirement (2.25b) and the normalizing condition (2.24a). The result is

$$\mu = \frac{(a_i f_{i,\mu k} a_j a_k)}{2(a_i f_{i,jk} a_j \dot{t}_{Ik} + a_i f_{i,j0} a_j \dot{\rho}_I)}$$
$$\alpha = |[\mu\dot{\rho}_I, \mu\dot{\bar{t}}_I + \bar{a}]|^{-1} = \{\mu^2 + 2\mu\bar{a}\dot{\bar{t}}_I + 1\}^{-1/2}. \tag{2.37}$$

It is finally noted, that a detailed derivation of the results shown in this section can be found in the appendix of [16]. Only the presentation of the result (2.36) is modified with respect to the one shown in [16]. This is done here to allow for the inclusion of the special case: $a_i f_{i,jk} a_j a_k = 0$.

3. THE INCREMENTAL SOLUTION OF EQUILIBRIUM PATHS

3.1 *General strategy*

In general, eqns (2.3) admit a number of solutions $\bar{t}$ for a given value of the load ρ but usually only a few of these solutions are important to the stability analysis. For the time being, attention will be focused to solutions $\bar{t}$ which correspond to a continuous deformation of the structure from the undeformed state. Equilibrium states thus obtained belong to the so-called basic or fundamental equilibrium path of the structure under consideration.

Iteration methods for the solution of nonlinear equations like (2.3) require, as is well known, a starting configuration (or initial iterate) which is "close" to the solution to be determined. This requirement fits well into the scenario of an incremental procedure which is designed to solve an equilibrium path in terms of a sequence of successive, but distinct points. Each point obtained offers a means to construct the starting configuration for the next to be computed. The accuracy of the "initial" iterate can be controlled by keeping the distance between the known and the still unknown point within certain bounds.

Two well known strategies are depicted in Figs. 2(a) and 2(b). In the first case, the load parameter ρ is used as the prescribed variable (denoted by η). In the second case, one of the deformation parameters t_K is taken to fulfill this role. Each point computed by the first method, is determined by the intersection of a surface ρ = constant (= η) and the equilibrium path governed by eqns (2.3). A point computed by the second method, is determined by the intersection of a surface t_K = constant (= η) with the same solution curve. This explains why both methods (a) and (b) break down in the neighbourhood of the turning points in Figs. 2(c) and 2(d) respectively.

The points in question are characterised by

$$\dot{\rho} = \frac{d\rho}{ds} = 0 \quad \text{for} \quad \rho = \eta^*$$

†There are more solutions when the bifurcation point is no longer distinct.

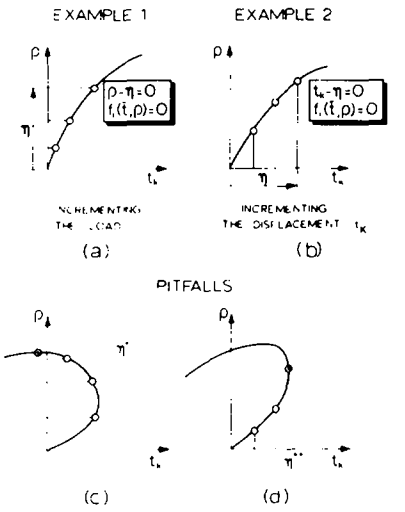

Fig. 2. Examples incremental solution methods.

and

$$\dot{t}_K = \frac{dt_K}{ds} = 0 \quad \text{for} \quad t_K = \eta^{**} \tag{3.1}$$

where it is noted, that the first definition given corresponds to the condition for a limit point. In either case, incrementing the value of η beyond the limiting values (3.1) leads to specifications of the surfaces

$$f_0 = \rho - \eta = 0$$

$$f_0 = t_K - \eta = 0$$

which do not define intersections with the equilibrium path, at least not in the neighbourhood of the points defined by (3.1). The breakdown of the procedures described does usually not occur suddenly, but is announced some time in advance by a marked increase in number of iterations necessary to obtain converged solutions. There is some evidence[14, 15], that this phenomena is coupled to the decrease in quality of the intersection of the surfaces (3.2) with the equilibrium path† when the critical points (3.1) are approached.

A measure of the quality of intersection is given by θ, the angle between the tangent of the equilibrium curve and the normal to the intersecting surface at the point of intersection. The intersection is considered good if θ is close or equal to zero and bad if it is close or equal to $\pi/2$. In this sense, ideal would be a family of surfaces (defined for a range of parameter values η)

$$f_0(\bar{t}; \rho; \eta) = 0 \tag{3.3}$$

which intersects the equilibrium curve everywhere according to the condition $\theta = 0$ (Fig. 3). Of

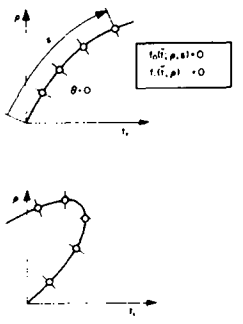

Fig. 3. Ideal procedure.

†In[14, 15], this connection is not properly formulated. However, it does not invalidate the finding.

course, it is not possible to construct such a set of "ideal" surfaces, because this would require the advance knowledge of the solution curve defined by (2.3). However, it is not necessary to insist on $\theta = 0$ at the points to be computed. Also a small value ($\theta < \pi/2$) would be sufficient to protect the procedure against failures of the type described. The relaxation of the condition $\theta = 0$ to $\theta < \pi/2$ opens a whole world of possible formulations of eqns (3.3). The particular choice made in this paper will be introduced now.

Let $[\rho_1, \bar{t}_1]$ and $[\dot{\rho}_1, \dot{\bar{t}}_1]$ denote a point of the path and its corresponding tangent respectively, and let η_1 denote the value of the path or control parameter η at this point. Suppose that these vectors have been computed at some stage of the computational process and that the next point is about to follow. The equation

$$f_0(\bar{t}; \rho; \eta) = \dot{\rho}_1(\rho - \rho_1) + \dot{\bar{t}}_1(\bar{t} - \bar{t}_1) - (\eta - \eta_1) = 0 \tag{3.4}$$

defines a surface, i.e. a plane, with $[\rho, \bar{t}]$ denoting the corrdinates of an arbitrary point of the plane. This plane is normal to the tangent $[\dot{\rho}_1, \dot{\bar{t}}_1]$ and its distance to $[\rho_1, \bar{t}_1]$ is $(\eta - \eta_1)$. It will be intersected by the equilibrium curve with a small angle θ if the distance $(\eta - \eta_1)$ is kept small. As soon as the point of intersection is computed for some value of η, eqn (3.4) can be redefined, which means that $[\rho_1, \bar{t}_1]$, $[\dot{\rho}_1, \dot{\bar{t}}_1]$ in (3.4) are replaced by the values that correspond to the new point obtained. It becomes clear, that in this way control surfaces are created, which comply very well with the requirement of a small θ along the path.

The control parameter η as defined by (3.4) and the computational procedure based on this choice, were first introduced in[14]. It is noted that differentiation of (3.4) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(f_0) = \{\dot{\rho}(\eta_1)\dot{\rho}(\eta) + \dot{t}_h(\eta_1)\dot{t}_h(\eta)\}\left(\frac{\mathrm{d}s}{\mathrm{d}\eta}\right) - 1 = 0 \tag{3.5}$$

where, as before, the dot is used to indicate differentiation with respect to s, the arc length of the solution curve. It can be deduced from this expression (see also 2.24), that for $\eta_1 \to \eta$ the ratio $(\mathrm{d}s/\mathrm{d}\eta)$ approaches unity. This means that η introduced by (3.4) serves as an approximation of s for small values of $(\eta - \eta_1)$. Therefore, in the following the symbol η will be replaced by s to indicate the geometrical significance of the present choice of the control parameter. The context in which s appears will usually be sufficient to explain whether this parameter is used in the exact or in the approximate sense.

3.2 *Newton's method*

For what follows it is convenient to introduce a more compact notation. Since the load parameter ρ and the deformation parameters t_h will be treated as unknowns on an equal basis, a configuration $[\rho, \bar{t}]$ can better be written as

$$\begin{aligned} &\bar{x} = x_0\bar{e}_0 + x_h\bar{e}_h = x_\alpha\bar{e}_\alpha \\ &x_0 = \rho;\ x_h = t_h (h = 1, 2, 3, \ldots . N). \end{aligned} \tag{3.6}$$

The control parameter s is introduced by adding the equation $f_0 = 0$ as a constraint condition to equation $f_i = 0$. Consequently, one can write

$$f_\alpha(\bar{x}; s) = 0 \qquad (\alpha = 0, 1, 2, \ldots . N). \tag{3.7}$$

With this, the convention is introduced that Greek indices cover the range (0, 1,....,N), while Latin indices remain reserved to cover the range $(1, 2, \ldots, N)$. It is noted that in agreement with the foregoing, partial differentiation with respect to x_α is denoted by

$$\begin{aligned} &\frac{\partial}{\partial x_0} = \frac{\partial}{\partial \rho} = (\ )_{,0};\ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t_i} = (\ )_{,i} \to \frac{\partial}{\partial x_\alpha} = (\ )_{,\alpha} \\ &(\alpha = 0, 1, 2, \ldots N). \end{aligned} \tag{3.8}$$

Equation (3.4) in this notation is

$$\dot{x}_\alpha(s_1)\{x_\alpha(s) - x_\alpha(s_1)\} - (s - s_1) = 0$$

or

$$\dot{\bar{x}}(s_1)\{\bar{x}(s) - \bar{x}(s_1)\} - (s - s_1) = 0. \tag{3.9}$$

It is expedient to make a distinction between exact or converged solutions of eqns (3.7) and approximate solutions or iterates. This will be done by the following convention

$$\begin{aligned} &\bar{x} = (x_0, x_1, x_2, \ldots . x_N) && \text{(exact)} \\ &\bar{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \ldots . \sigma_N) && \text{(approximate solution or iterate)} \\ &\bar{\tau} = (\tau_0, \tau_1, \tau_2, \ldots . \tau_N) && \text{(improved approximation or higher order iterate).} \end{aligned} \tag{3.10}$$

Applied to eqns (3.7), the method of Newton[25] is defined by the system of equations

$$f_{\alpha,\beta}(\bar{\sigma};s)\Delta\sigma_\beta + f_\alpha(\bar{\sigma};s) = 0 \tag{3.11}$$

with

$$\tau_\alpha = \sigma_\alpha + \Delta\sigma_\alpha. \tag{3.12}$$

It can be defined in inverse form

$$\tau_\alpha = \sigma_\alpha - F_{\alpha,\beta}(\bar{\sigma};s)f_\beta(\bar{\sigma};s) \tag{3.13}$$

where $F_{\alpha,\beta}$ correspond to the definition

$$[F_{\alpha,\beta}] = [f_{\alpha,\beta}]^{-1}. \tag{3.14}$$

A configuration $\bar{x}(s)$ defined by eqns (3.7) is computed as follows. An approximation $\bar{\sigma}$ is entered into (3.11), the system is solved for $\Delta\bar{\sigma}$ and an improved solution $\bar{\tau}$ results, (3.12). The process is repeated and convergence is obtained if $\bar{\sigma}$ approaches $\bar{x}$ sufficiently close. The literature about the method is extensive so that a detailed description can be omitted (for instance[25–27]). Three global points should be mentioned however: (a) The method makes use of the inverse of $[f_{\alpha,\beta}]$ which means that the Jacobian must satisfy

$$J = \det\{f_{\alpha,\beta}\} \neq 0.$$

(b) If $J \neq 0$, a finite domain $|\bar{\sigma} - \bar{x}|^2 = \delta^2(\bar{x})$ around the point $\bar{x}$ can be defined so that any initial iterate $\bar{\sigma} = \bar{\sigma}^{(1)}$ which satisfies $|\bar{\sigma} - \bar{x}|^2 < \delta^2(\bar{x})$ leads to a convergent process. (c) The rate of convergence is quadratic, i.e.

$$|\bar{\tau} - \bar{x}| = L|\bar{\sigma} - \bar{x}|^2$$

where L is some positive number which depends on the configuration $\bar{x}(s)$.

The effectiveness of the procedure to be discussed now is for a great deal based on these properties.

3.3 *Basic procedure*

Suppose that of the equilibrium path to be computed, one point is known in advance together with its path derivative, and denote these vectors by

$$\begin{aligned} \bar{x}_1 &= \bar{x}(s_1) \\ \dot{\bar{x}}_1 &= \dot{\bar{x}}(s_1). \end{aligned} \tag{3.15}$$

An estimate of a next point of the path is then readily given by

$$\bar{\sigma}^{(1)} = \bar{x}_1 + \Delta s\dot{\bar{x}}_1 = \bar{x}_1 + (s - s_1)\dot{\bar{x}}_1. \tag{3.16}$$

This approximate solution $\tilde{\sigma}^{(1)}$ can be improved by the successive solution of

$$f_{\alpha,\beta}(\tilde{\sigma}^{(i)})\Delta\sigma_\beta^{(i)} + f_\alpha(\tilde{\sigma}^{(i)};\ s_1 + \Delta s) = 0$$
$$\tilde{\sigma}^{(i+1)} = \tilde{\sigma}^{(i)} + \Delta\tilde{\sigma}^{(i)} \tag{3.17}$$
$$i = 1, 2, 3, \ldots M$$

until for $i = M$, a sufficiently accurate solution is obtained. (The question how this accuracy can be controlled effectively is not a part of the discussion at the present time. The way it is usually done is mentioned in Section 5.1.) A picture of the procedure defined by (3.16) and (3.17) is given in Fig. 4.

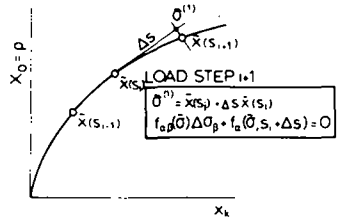

Fig. 4. Basic procedure.

For the computation of the next point of the path $(s + \Delta s)$ it is necessary to compute the tangent $\dot{\tilde{x}}(s)$. It can be done by means of the set of equations

$$f_{\alpha,\beta}(\tilde{x} = \tilde{\sigma}^{(M)})b_\beta - \delta_{\alpha 0} = 0 \tag{3.18a}$$

$$\dot{\tilde{x}}(s) = \frac{\tilde{b}}{|\tilde{b}|}. \tag{3.18b}$$

Equation (3.18a) follow from eqns (3.7) when the latter are differentiated with respect to $s(b_\alpha = \mathrm{d}x_\alpha/\mathrm{d}s)$. The normalizing condition (3.18b) is a consequence of the circumstance that the parameter s in (3.7) can only be appreciated as an approximation to the arc length of the path. It goes without saying that one can quite well do away with the "exact" value of the tangent vector $\dot{\tilde{x}}$ as determined by (3.18) and use an approximation instead. A simple example of such alternative is

$$\dot{\tilde{x}}(s) \cong \frac{\dot{\tilde{x}} - \dot{\tilde{x}}_1}{|\dot{\tilde{x}} - \dot{\tilde{x}}_1|}. \tag{3.19}$$

The choice depends somewhat on the way the complete procedure is implemented. However, the first possibility should always be preferred in cases where the extra volume of operations connected with (3.18) can be accepted.

3.4 *Justification of the method*

The effectiveness of Newton's method as it was formulated in Section 3.2 depends in the first place on the existence of a non-singular Jacobian $[f_{\alpha,\beta}]$. In the second place, successful application of the method depends on the "closeness" of the initial iterate $\tilde{\sigma}^{(1)}$ to the true solution $\tilde{x}$.

The first requirement concerns

$$J = \det[f_{\alpha,\beta}]. \tag{3.20}$$

Expansion of this determinant with respect of the first row gives

$$J = f_{0,0}D + f_{0,i}A_i$$

where A_i are the co-factors of $f_{0,i}$ in J. It can easily be verified that the co-factors A_i must be

identical to the vector components d_i introduced in Section 2. Thus,

$$J = f_{0,0}D + f_{0,i}d_i \qquad [14, 15]. \tag{3.21}$$

With the present choice of f_0, eqn (3.4), this expression can be written in the form

$$J(s) = \dot{x}_0(s_1)d_0(s) + \dot{x}_i(s_1)s_i(s) = \dot{x}_\alpha(s_1)d_\alpha(s) \tag{3.22}$$

where $d_0(s)$ is used to denote the determinant $D(s)$ for convenience. The parameter value s corresponds to the intersection of $f_0 = 0$ and $f_i = 0$, while s_1 corresponds to a previous point of the solution curve in agreement with the foregoing notation. Although evaluated at the point $\tilde{x}(s)$, $J(s) = \det\{f_{\alpha,\beta}(\tilde{x})\}$ is also dependent on the point $\tilde{x}(s_1)$ through the derivative $\dot{\tilde{x}}(s_1)$.

In the compact notation introduced earlier, eqns (2.27) read

$$d_0(s)\dot{x}_\alpha(s) = d_\alpha(s)\dot{x}_0(s). \tag{3.23}$$

The square of the "length" of both sides leads to the equality

$$d_0^2(s) = \dot{x}_0^2(s)\{d_\alpha(s)d_\alpha(s)\}. \tag{3.24}$$

The scalar product of both sides of (3.23) with $\dot{x}_\alpha(s_1)$ and use of (3.21) results in

$$d_0(s)\{\dot{x}_\alpha(s_1)\dot{x}_\alpha(s)\} = \dot{x}_0(s)J(s). \tag{3.25}$$

Equations (3.24) and (3.25) can be combined to give

$$J^2(s) = \{\dot{x}_\alpha(s_1)\dot{x}_\alpha(s)\}^2\{d_\alpha(s)d_\alpha(s)\}. \tag{3.26}$$

It is noted that there are two possibilities for $J(s)$ to become zero. The first occurs if the angle θ between $\dot{\tilde{x}}(s)$ and $\dot{\tilde{x}}(s_1)$ is equal to $\pi/2$, because then

$$\cos\theta = \dot{\tilde{x}}(s_1)\dot{\tilde{x}}(s) = 0. \tag{3.27}$$

It is clear that this situation can easily be avoided by taking the stepsize $(s - s_1)$ small.

The second possibility occurs when

$$\tilde{d} = \tilde{0} \text{ or } D = 0;\ d_i = 0 \tag{3.28}$$

which is the condition that is satisfied at a bifurcation point. Consequently, $[f_{\alpha,\beta}]$ will be non-singular along the path if bifurcation points are excluded and if the steps $\Delta s = s - s_1$ are taken small enough.

The second requirement for convergence of Newton's method concerns the distance of the initial iterate $\tilde{\sigma}^{(1)}$ to the exact solution $\tilde{x}(s)$. It must be clear that this distance can be made as small as one wishes because the use of the predictor (3.16) implies

$$|\tilde{\sigma}^{(1)} - \tilde{x}| \leqq \frac{1}{2}|\ddot{\tilde{x}}(s)|\Delta s^2 + |0(\Delta s^3)|. \tag{3.29}$$

Therefore, the procedure described in the previous section should be effective along any smooth equilibrium path governed by eqns (2.3), as long as the stepsize Δs is kept small enough and as long as no attempt is made to compute a bifurcation point or a point in a close neighbourhood thereof. A rigorous account of this method has recently been given in[32]; see also Section 5.4. It is finally noted, that eqn (3.25) can written as

$$D(s) = \dot{\rho}(s)J(s)\{\dot{\tilde{x}}(s_1)\dot{\tilde{x}}(s)\}^{-1} \tag{3.30}$$

which illustrates with more clarity the connection between the stability determinant D and the determinant J of the extended system of equations. Relation (3.30) is useful for the detection of critical points as will be seen later.

4. APPLICATIONS

4.1 *Determination of a critical state*

An important part of the stability analysis is the determination of the critical state. In the case of a snapping problem, the computation of the critical state (the limit point) is usually the final goal of the analysis. In the case of a buckling problem the determination of the bifurcation state is the point of departure for an evaluation of the specific properties of this state.

Conditions for the occurrence of critical points were stated in Section 2. The lowest eigenvalue of $[f_{i,j}]$ should satisfy

$$\omega_1\{(\bar{x}(s)\} = 0.$$

It is also possible however to detect a critical point by the vanishing of the determinant $D\{\bar{x}(s)\}$, thus

$$D\{\bar{x}(s)\} = 0.$$

It is feasible to determine the critical state of the basic path using a strategy which can be described as follows. Compute points along the path $\bar{x}(s_i)$ together with the values $\omega(s_i)$ or $D(s_i)$ until the quantities last mentioned undergo a change of sign†. Construct polynomials through $\bar{x}(s_i)$ and $\omega_1(s_i)$ or $D(s_i)$ and construct with use of these polynomials the approximation of $\bar{x}(s^*)$ for which the zero of ω_1 or D is attained. It is known that procedures of this nature have been used successfully[20, 21]. However, an unpleasant aspect of this method is the uncertainty about the stepsize Δs to be used. The stepsize Δs is determinate for the number of points computed before a zero of ω_1, or D will be encountered. Too many points are wasteful from an economical point of view, too few points will lead to inaccurate results. Therefore, at present a strategy is adopted which guides the computation in the direction of the critical point. The aim is to reduce the number of intermediate results (the non-critical points) as much as possible.

Suppose that a sequence of pre-critical points of the basic path have been computed and denote these points by

$$\bar{x}(s_1); \bar{x}(s_2); \ldots\ldots; \ldots\ldots \bar{x}(s_i) \tag{4.1}$$

$$0 < s_1 < s_2 < \cdots\cdots < s_i.$$

Suppose further that the stability coefficients, i.e. the eigenvalue $\omega_1(s)$ and the determinant $D(s)$ can be expanded according to the series

$$\omega_1(s) = \omega_1(s_i) + \dot{\omega}_1(s_i)(s - s_i) + \frac{1}{2}\ddot{\omega}_1(s_i)(s - s_1) + \ldots$$

$$D(s) = D(s_i) + \dot{D}(s_i)(s - s_i) + \frac{1}{2}\ddot{D}(s_i)(s - s_i)^2 + \ldots. \tag{4.2}$$

where s_i corresponds to the last point of the sequence (4.1). Within a certain neighbourhood of the critical state, the slopes $\dot{\omega}_1(s)$, $\dot{D}(s)$ must be negative‡. Suppose now that $\bar{x}(s_i)$ is within this neighbourhood. It then follows that a first estimate of the critical state $\bar{x}^* = \bar{x}(s^*)$ (relative to the point $\bar{x}(s_i)$) can be given by

$$\bar{\sigma}^* = \bar{x}(s_i) + \Delta s^* \dot{\bar{x}}(s_i) \tag{4.3}$$

†Note, that at this point the notation $\omega(k)$ is changed to ω_k.
‡See also the first remark of Section 5.3.

where Δs is determined by either

$$\Delta s^* = (s^* - s_i) = -\frac{\omega_1(s_i)}{\dot{\omega}_1(s_i)} \tag{4.4a}$$

or

$$\Delta s^* = (s^* - s_i) = -\frac{D(s_i)}{\dot{D}(s_i)}. \tag{4.4b}$$

Postpone the examination into the possibilities to compute (4.4a) or (4.4b) for the time being and assume that one of these estimates has been obtained.

If the approximation (4.3) is not satisfactory, a new estimate should be constructed relative to a point which is nearer to the critical state. Such a point can conveniently be computed with the basic procedure in which the predictor (3.16) is modified to

$$\tilde{\sigma}^{(i)} = \bar{x}(s_i) + \eta \Delta s^* \dot{\bar{x}}(s_i). \tag{4.5}$$

In this expression η is some measure to serve as a means to keep the stepsize $\Delta s = \eta \Delta s^*$ within acceptable bounds. After the new approximation (4.3) has been obtained the process can be repeated and so on. The complete procedure is convergent if Δs^* approaches zero in the subsequent steps. Figure 5 shows a picture of such a procedure.

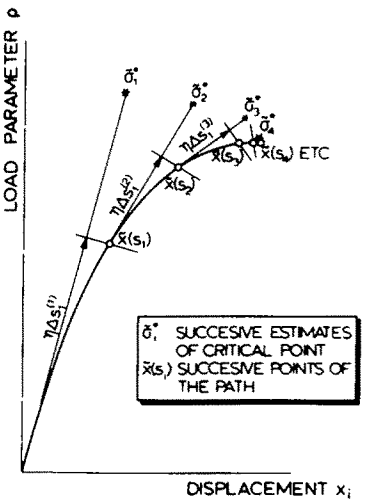

Fig. 5. Computation of critical point.

It does not seem possible to give a recipe for the choice of the factor η. If one insists on approaching the critical state from below, i.e. from the stable part of the path (as was tacitly assumed in the foregoing), it should be given the value $0 < \eta \leqq 1$ in most cases; with $\eta = 1$ probably as one of the best choices. Much depends on the accuracy of the estimate Δs^* and the error of the first iterate $\tilde{\sigma}^{(1)}(\tilde{\delta} = \bar{x}(s_i) + \eta \Delta s^* \dot{\bar{x}}(s_i) - \bar{x}(s_{i+1}))$ which should be within a certain bound as was discussed earlier. (The strategy whereby the critical state is reached from below is a convenient one from a programming point of view but it is by no means necessary. This particular aspect of the implementation of the procedure will not be discussed in this paper).

The time has come to deal with the question how the estimates (4.4a) and (4.4b) can be computed.

(a) *The estimate* (4.4*a*). The characteristic value $\omega_1(s)$ of $[f_{i,j}]$ is determined by eqns (2.13)

$$f_{i,j}\{\bar{x}(s)\}a_j - \omega_1(s) T_{ij} a_j = 0 \tag{2.13}$$

where T_{ij} is some suitably chosen, positive definite reference matrix not considered dependent on s. Note that $\bar{a}$ is written for $\bar{a}(1)$.

These equations are valid along the path for any s. Consequently, it is permissible to

differentiate (2.13) with respect to s, which gives

$$f_{i,j}\dot{a}_j - \omega_1 T_{ij}\dot{a}_j + \dot{f}_{i,j}a_j - \dot{\omega}_1 T_{ij}a_j = 0. \tag{4.6}$$

By taking the scalar product of $\tilde{a}$ with the expression on the l.h.s. of (2.13) and (4.6) one can show that

$$\omega_1(s) = \frac{f_{i,j}a_i a_j}{T_{ij}a_i a_j}$$

$$\dot{\omega}_1(s) = \frac{\dot{f}_{i,j}a_i a_j}{T_{ij}a_i a_j} \tag{4.7}$$

where

$$\dot{f}_{i,j} = \frac{\mathrm{d}}{\mathrm{d}s}(f_{i,j}) = f_{i,ij} = f_{i,j\alpha}\dot{x}_\alpha. \tag{4.8}$$

Thus (4.4a) can be computed if ω_1, $\tilde{a}$, $\dot{\tilde{x}}$ and $[\dot{f}_{i,j}]$ are known at the point $\tilde{x}_i$. The first eigenvalue and corresponding vector can be obtained from (2.13) by standard procedures. The slope $\dot{\tilde{x}}(s_i)$ at the point $\tilde{x}(s_i)$ is automatically produced when the procedure of Section 3.3 is used. However, the computation of $[\dot{f}_{i,j}]$ needs more consideration. The r.h.s. of (4.8) looks straightforward but nevertheless necessitates the evaluation of the third order derivatives $f_{i,j\alpha}$† which occupy an array of dimension $N^2(N+1)$. Operation with such a large set of functions is cumbersome and will soon lead to insurmountable storage problems. Therefore, in this paper the point of view is taken that the use of derivatives of order three and higher should be avoided if it is possible to do so. A possible way to circumvent this difficulty is to use an approximation, e.g.

$$\dot{f}_{i,j}(s_i) \cong \frac{f_{i,j}(s_i) - f_{i,j}(s_{i-1})}{(s_i - s_{i-1})}. \tag{4.9}$$

The disadvantage here is that $[f_{i,j}(s_{i-1})]$ must be stored and to a lesser extent that at least two points of the path must be determined before it can be used. Yet another possibility to approximate $\dot{\omega}(s_i)$ is

$$\dot{\omega}(s_i) \cong \frac{\omega(s_i) - \omega(s_{i-1})}{(s_i - s_{i-1})} \tag{4.10}$$

which is the simplest of all, but which needs two evaluations of ω before it can be applied.

(b) *The estimate* (4.4*b*). Application of (4.4a) requires the evaluation of the smallest eigenvalue of $[f_{i,j}]$ (in absolute sense), which adds considerably to the volume of the computations, at least when standard routines are used. It might well be that the second estimate (4.4b) is a more economical alternative. The evaluation of the derivative $\dot{D}(s)$ in an exact sense is impractical for large systems of equations because it requires too many numerical operations. Therefore, also in this case, finite difference approximations are a possible way out. The simplest choice is

$$\dot{D}(s_i) \cong \frac{D(s_i) - D(s_{i-1})}{(s_i - s_{i-1})}. \tag{4.11}$$

Clearly, (4.11) can only be used with advantage if the determinant values $D(s_i)$ can be obtained in a relatively easy way. If the solution of eqns (3.17), a decomposition method is used, the determinant $J = \det\{f_{\alpha,\beta}\}$ will automatically be provided. In that case the computation of (4.11) is straightforward because, see eqn (3.30)

$$D(s_{i-1}) = \dot{\rho}(s_{i-1})J(s_{i-1})\{\dot{x}_\alpha(s_{i-2})\dot{x}_\alpha(s_{i-1})\}^{-1}$$

$$D(s_i) = \dot{\rho}(s_i)J(s_i)\{\dot{x}_\alpha(s_{i-1})\dot{x}_\alpha(s_i)\}^{-1}. \tag{4.12}$$

†Third order derivatives of the potential energy P.

It is noted that eqns (4.12) require three points of the path to be known in advance which follows from the appearance of the three different tangents in these expressions. If this property is undesirable, one can drop the terms $[\dot{x}_\alpha(s_{i-2})\ \dot{x}_\alpha(s_{i-1})]$ and $[\dot{x}_\alpha(s_{i-1})\ \dot{x}_\alpha(s_i)]$. This follows because the zero's of $D(s)$ are essentially contained in the zero's of $\dot{\rho}(s)$ or $J(s)$†.

4.2 *Continuation beyond the critical points*

It is often desirable to extend the computation of the basic equilibrium path beyond the critical states. It must be clear that this task can be carried out with the procedure discussed in Section 3. As was shown in Section 3.4, the basic incremental method used here is only sensitive to a bifurcation point because there the Jacobian matrix $[f_{\alpha,\beta}]$ of eqns (3.17) becomes singular. Passing a limit point or a turning point with respect to any other variable t_K does therefore not pose a problem. Passing a bifurcation point need not be a problem too, as long as the use of eqns (3.17) close to or at this particular point can be avoided. In fact, when the stepsize Δs is chosen arbitrarily there is little chance that this will occur. Consequently, continuation of the computation beyond limit and bifurcation points (Fig. 6), can be carried out without special or difficult measures in the implementation of the procedure.

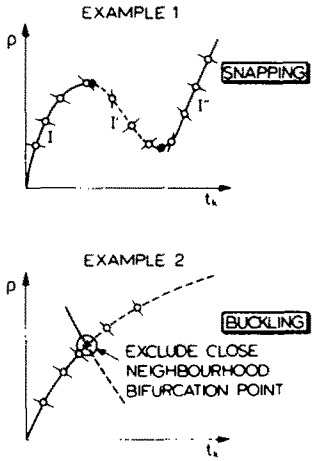

Fig. 6. Continuation along basic path.

4.3 *Continuation onto a branch*

The discussion is slightly more involved in cases where one wishes to compute the branch of a bifurcation point. It will be assumed here, that the bifurcation point together with its tangent to the basic path are obtained beforehand, e.g. as a result of the method discussed in Section 4.1. They will be denoted by $\bar{x}(s^*)$, $\dot{\bar{x}}_I(s^*)$ respectively.

As should have become clear from the foregoing, continuation along any smooth path is possible as long as at least one regular point of the path is known in advance. In that case, the tangent to the path can be obtained from eqn (3.18) and the subsequent operations can be carried out in a straight-forward manner, i.e. in the way it is described in Section 3.3. However, of the branch of the basic path, only the configuration $\bar{x}(s^*)$ is known in advance and this state corresponds to a singular matrix $[f_{\alpha,\beta}]$. The tangent to the branch at this point cannot be computed from (3.18), but follows from eqns (2.37). In the shortened notation they read

$$\dot{\bar{x}}_I = \frac{\bar{a} + \mu\dot{\bar{x}}_I(s^*)}{|\bar{a} + \mu\dot{\bar{x}}_I(s^*)|}\ddagger$$

$$\mu = -\frac{a_i f_{i,j\alpha} a_j a_k}{2a_i f_{i,j\alpha} a_j \dot{x}_{k\alpha}(s^*)}. \tag{4.13}$$

†One should expect loss of accuracy of (4.11) in that case.
‡Note that it is implied that $\bar{a} = [0, a_1, a_2, \ldots, a_N]$.

Continuation onto the branch can now be started with the following initial configuration

$$\bar{\sigma}^{(1)} = \bar{x}(s^*) + \Delta s \dot{\bar{x}}_{II}(s^*). \tag{4.14}$$

Unfortunately the calculation of $\dot{\bar{x}}_{II}(s^*)$ from expressions (4.13) involves the evaluation of the derivatives $f_{i,j\alpha}$ which is, as was explained before, not attractive from a practical point of view. It was shown in[16], that it is possible to compute μ in (4.13) without the use of these functions. However, it was also demonstrated that another start of the procedure onto the branch can be made which abandons the direction vector (4.13) altogether.

First it is noted that the tangent $\dot{\bar{x}}_I$ is not a necessary choice for $f_{0,\alpha}$ in the formulation of the auxiliary surface $f_0 = 0$. The computation of the set of eqns $f_\alpha = 0$, (3.7), can be carried out with any other normal vector $[f_{0,\alpha}]$ as long as intersection of the plane $f_0 = 0$ with the path remains reasonably well preserved. With this in mind, an effective choice for the normal $[f_{0,\alpha}]$ is given by

$$f_{0,\alpha} = \frac{\dot{x}_{I\alpha}(s^*) + \mu_1 a_\alpha}{|\dot{x}_{I\alpha}(s^*) + \mu_1 a_\alpha|} \tag{4.15a}$$

where the factor μ_1 is determined by the product

$$\mu_1 = -\{\dot{\bar{x}}_I(s^*)\bar{a}\}^{-1}. \tag{4.15b}$$

It follows by inspection, that the scalar product of $\dot{x}_I$ and $\dot{x}_{II}$ with $[f_{0,\alpha}]$ gives

$$\dot{x}_{I\alpha}(s^*) f_{0,\alpha} = 0$$

$$\dot{x}_{II\alpha}(s^*) f_{0,\alpha} \neq 0.$$

Thus the surface $f_0 = 0$ defined by the normal (4.15) is parallel to the tangent $\dot{\bar{x}}_I(s^*)$. This implies that in the neighbourhood of the critical state intersections of the basic path with the surface $f_0 = 0$ cannot occur. In other words, the computational process is prevented to return to the basic state. The predictor in the first step onto the path is now given by

$$\bar{\sigma}^{(1)} = \bar{x}(s^*) + \Delta s f_{0,\alpha} \bar{e}_\alpha. \tag{4.16}$$

Once the point corresponding to (4.16) is determined, the tangent to the branch at this point can be computed with (3.18) or (3.19) and continuation of the procedure can further be carried out in the manner described before. The advantage of the predictor (4.16) over the one discussed earlier, (4.14), is the simplicity of its construction. Only the explicit knowledge of $\bar{a}$ the first eigenvector of $[f_{i,j}(s^*)]$ is required. Possible failures as a result of the predictor (4.16) (determined by (4.15)) are anticipated when the angle between the basic path and branch is small at the critical state, thus when

$$\cos\theta = [\dot{\bar{x}}_I(s^*) \cdot \dot{\bar{x}}_{II}(s^*)] \sim 1.$$

In that case one is forced to attempt the alternatives mentioned earlier.

5. EXAMPLES AND CONCLUSION

5.1 *Shallow circular arch*

The stability of a circular arch, uniformly loaded in radial direction, is lost in either a limit point or a bifurcation point depending on the dimensions of the arch and the boundary conditions enforced at the ends[27, 28]. This property offers the opportunity to test the numerical techniques discussed in the foregoing on one and the same model. In[16], a finite element method was employed for the discretization of the governing equilibrium equations. In this section, the results of the computations carried out in[16] will briefly be discussed.

Apart from the boundary conditions, the determining factor in the behaviour of the arch is

the parameter

$$K = \alpha^2 \frac{R}{h} \tag{5.1}$$

where R/h = radius to thickness ratio and 2α = arc-length of the arch. In the examples that follow K was set equal to either $K = 4$ or $K = 10$.

The first two examples in Figs. 7 and 8 respectively, represent two different calculations of

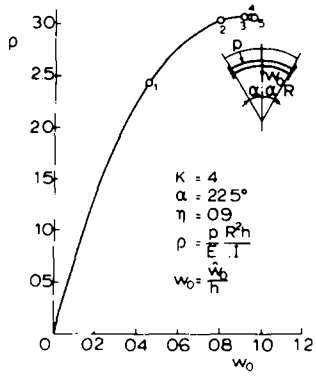

Fig. 7. Example limit point computation (I).

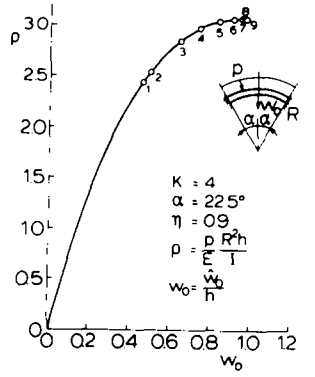

Fig. 8. Example limit point computation (II).

the same problem, i.e. the case "clamped–clamped; $K = 4$; $\alpha = 22.5°$". Procedure I made use of the predictions Δs^* of the type (4.4a) which was computed from the set of equations

$$[f_{i,j}\{\tilde{x}(s_k)\} + \Delta s \dot{f}_{i,j}\{\tilde{x}(s_k)\}]a_j = 0\dagger. \tag{5.2}$$

Procedure II was based on the predictor Δs^* of the type (4.4b), the approximate value of which is given by

$$\Delta s^* = -D(s_i)\frac{s_i - s_{i-1}}{D(s_i) - D(s_{i-1})}. \tag{5.3}$$

The direct way in which Δs^* is determined by (5.2) follows from the definition of (4.4a) together with (4.7) if T_{ij} in (2.13) is taken to be

$$T_{ij} = -\dot{f}_{i,j}\{\tilde{x}(s_k)\}_{s_k = \text{fixed}}. \tag{5.4}$$

More details on this particular choice of T_{ij} can be found in [16].

The iterative improvements of each solution point were terminated after the improvement $\Delta\tilde{\sigma}^{(i)}$ satisfied

$$|\Delta\tilde{\sigma}^{(i)}| < r_1 = \epsilon_1 \Delta s^{*(1)} \tag{5.5}$$

where ϵ_1 is some small, preassigned, positive number. The value $\Delta s^{*(1)}$ corresponded to the first estimate

$$\{f_{i,j}(\tilde{0}) + \Delta s \dot{f}_{i,j}(\tilde{0})\}a_j = 0. \tag{5.6}$$

The approximation of the critical state

$$\tilde{\sigma}^* = \tilde{x}(s_i) + \Delta s^{*(i)}\dot{\tilde{x}}(s_i) \tag{5.7}$$

was considered to be adequate if the prediction $\Delta s^{*(i)}$ in the step (i) satisfied

$$|\Delta s^{*(i)}| < r_2 = \epsilon_2 \Delta s^{*(1)} \tag{5.8}$$

again with ϵ_2 preassigned.

†In this particular case $\omega_1 = \Delta s^*$; $\dot{\omega}_1 = 1$, [16].

Table 1.

STEP	PROCEDURE I			PROCEDURE II		
	p	w_0 10^{-1}	ITERATION COUNT	p	w_0 10^{-1}	ITERATION COUNT
1	2.4354	4.6033	2	2.43534	4.6033	2
2	3.0290	8.0578	2	2.5540	5.0246	1
3	3.0753	9.2763	2	2.8529	6.4525	1
4	3.0777	9.5919	1	2.9800	7.4538	1
5	3.0780	9.6413	0	3.0490	8.4085	1
6				3.0722	9.0871	1
7				3.0722	9.4694	1
8				3.0777	9.6074	1
9				3.0778	9.6401	0
ITERATION COUNT TOTAL			7	ITERATION COUNT TOTAL		9

The numerical results of the two runs are presented in Table 1. Measured in terms of the total number of iterations carried out with eqn (3.17), Procedure I is the most efficient of the two examined here.

However, the total computer run time used by a particular method also depends on other factors such as the time required to compute the estimates (5.2) and (5.3). A discussion about this important question and others connected with the economy of the methods proposed here is postponed to the next section.

Further examples of calculations carried out with Procedure I are given by Figs. 9–11. In none of these cases difficulties were encountered. Iterative improvement of the initial iterate $\tilde{\sigma}^{(1)}$ seldom exceeded 2 iterations if the termination condition (5.5) was applied with $\epsilon_1 = 0.005$. As expected, the largest number of iterations are needed at points which correspond to the largest step Δs.

An example of numerical continuation through a limit point is given in Fig. 12. An example of continuation onto a branch is presented in Fig. 13.

All these calculations were carried out on a system of equations of dimension 46, 47 or 50 depending on the boundary conditions used (a 16-element representation).

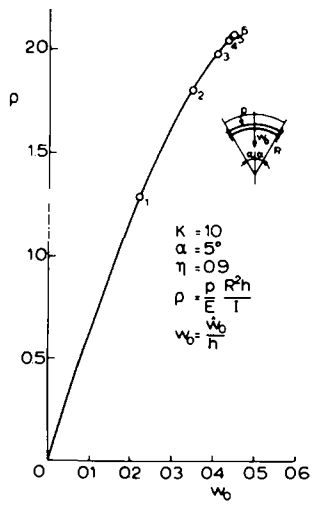

Fig. 9. Example computation bifurcation point (1).

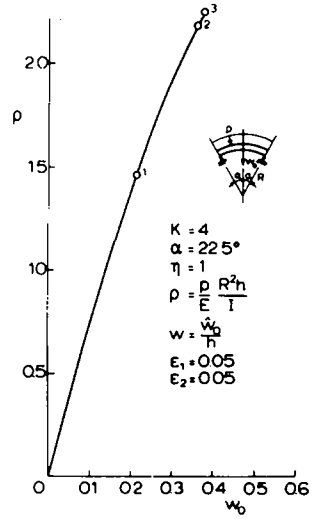

Fig. 10. Example computation bifurcation point (2).

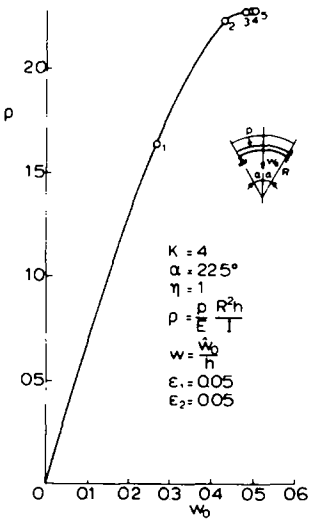

Fig. 11. Example computation limit point.

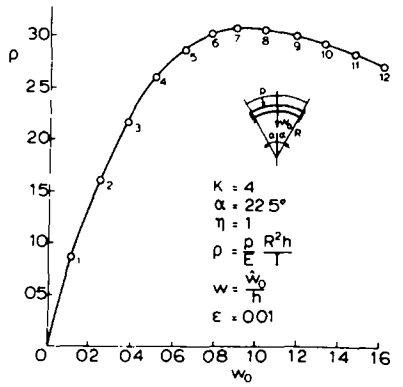

Fig. 12. Example continuation along basic path.

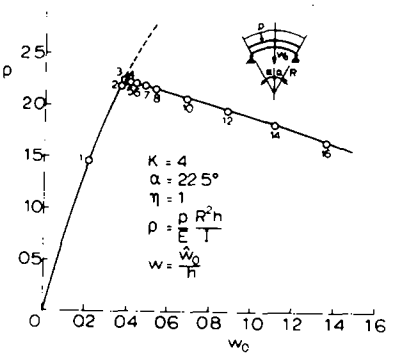

Fig. 13. Example continuation onto branch.

5.2 *Questions about efficiency*

Applied to the arch problem, the methods discussed in this paper prove to be effective in the sense that all equilibrium points relevant to the analysis of the arch can be obtained by them. However, as tools in the framework of a general purpose program for the stability analysis of structures, one desires next to effectiveness also an acceptable degree of efficiency of the methods employed.

The method of Newton on which the present procedures are based requires the repeated solution of the set of equations

$$f_{\alpha,\beta}(\tilde{\sigma}^{(i)})\Delta\sigma_\beta{}^{(i)} + f_\alpha(\tilde{\sigma}^{(i)}; s) = 0 \tag{5.9}$$

$$\tilde{\sigma}^{(i+1)} = \tilde{\sigma}^{(i)} + \Delta\tilde{\sigma}^{(i)}; \; i = 1, 2, \ldots . M$$

in order to obtain one particular solution $\tilde{\sigma}(s)$. The iteration count (M) varies from point to point in principle, and is strongly dependent on the accuracy of the predictor $\tilde{\sigma}^{(1)}$. The operations connected with the solution of (5.9) increase cubically with the dimension (N) of the problem at hand. It is expedient therefore to minimize the number of iterations with (5.9) as much as possible.

A strongly recommended way to deal with this problem is to improve the predictor (3.16) for instance by taking into account the curvature $\ddot{\tilde{x}}(s)$ of the path

$$\tilde{\sigma}^{(i)} = \tilde{x}(s_1) + \Delta s\dot{\tilde{x}}(s_1) + \left(\frac{\Delta s}{2}\right)^2 \ddot{\tilde{x}}(s_1). \tag{5.10}$$

Another improvement can be introduced by a relaxation of the termination condition for the Newton iteration cycle at points of the path which are not of particular interest (see [16] Section 6).

A different means to reduce the volume of the computations is based on a modification of Newton's method.

The idea is to reduce the computational effort connected with the factorization of $[f_{\alpha,\beta}]$†. It is well known that the iterative process defined by (5.9) will also be effective with a suitable approximation to the matrix $[f_{\alpha,\beta}(\tilde{\sigma})]$. For instance, in some neighbourhood of the point $\tilde{x}_1 = \tilde{x}(s_1)$, $[f_{\alpha,\beta}(\tilde{x}_1)]$ can be used in

$$f_{\alpha,\beta}(\tilde{x}_1)\Delta\sigma_\beta^{(i)} + f_\alpha(\tilde{\sigma}^{(i)}; s_1 + \Delta s) = 0$$
$$\tilde{x}(s) = \tilde{\sigma}^{(1)} + \sum_{i=1}^{M} \Delta\tilde{\sigma}^{(i)} \tag{5.11}$$

to compute points $\tilde{x}(s)$ for $s = s_1 + \Delta s$.

In this scheme, $[f_{\alpha,\beta}(\tilde{x}_1)]$ is factorized once. Each iteration involves only the computation of $[f_\alpha]$ and the construction of the improvement $\Delta\tilde{\sigma}$ (by back substitution). The effort spent is considerably less than the effort spent with a true Newton iteration. However, the modification does not converge quadratically [26] but linearly. Consequently the number of iterations required is larger than that required by Newton's method. The linear character of the method implies.

$$|\tilde{\sigma}^{(k)} - \tilde{x}| = L^{(k)}|\tilde{\sigma}^{(k-1)} - \tilde{x}| \tag{5.12}$$

where the factor $L^{(k)}$ depends on $\tilde{x}_1$ and $\tilde{x}(s)$.

It can be shown that the leading term of $L^{(k)}$ is linearly dependent on the distance $|\tilde{x}(s_i) - \tilde{x}(s_1)| \sim (s_i - s_1)$. For small $(s_i - s_1)$, $L^{(k)} < 1$ and convergence is assured. However, when $(s_i - s_1)$ increases, the rate of convergence will decrease until at some distance convergence breaks down altogether. Consequently, the method requires a redefinition of $[f_{\alpha,\beta}]$ from time to time but this will be required anyway if one wants to make use of the stability coefficients ω_1 or D.

The modification just described is used in the computer code STAGS [29, 30] which is a code to analyse the nonlinear behaviour of shells and plate structures. The authors report considerable improvement of the efficiency with respect to the original procedure based on Newton's method. In STAGS, the control parameter is either the load factor (ρ) or a multiplier connected with prescribed displacements. It will be interesting to see how this modification works when the length of the path is prescribed, because this choice of the control parameter seems to have a definite edge over other possible choices.

In the procedures for the computation of critical points the predictors based on (4.4a) or (4.4b) were introduced in order to reduce the number of intermediate results, that is the non-critical points $\tilde{x}(s_i)$. However, the use of these estimates can only be justified if it does not add considerably to the volume of the computations. When the solution of the linear system (5.9) or (5.11) is based on a decomposition method, the value of the determinant $J(\tilde{\sigma})$ will be produced as a consequence of this method. In that case, D, and therefore the estimate (4.4b), can easily be computed as is described in Section 4.1.

The situation is less simple for the estimate (4.4a). The calculation of ω_1 and the corresponding characteristic vector $\tilde{a}$ of the matrix $[f_{i,j}]$ is expensive if standard computer codes are used. However, improvement in economy is possible if more specialized methods are introduced. For instance, it is possible to obtain $\omega_1(s)$ and $\tilde{a}(s)$ from the decomposed $[f_{\alpha,\beta}]$, using an iterative cycle similar to that defined by (5.11). An evaluation of this possibility is documented in [33] and will be published in the near future.

5.3 *Concluding remarks*

(a) From the present description of the computation of critical points should not be deduced

†By factorization is meant the computation of $[f_{\alpha,\beta}(\tilde{\sigma})]$ and decomposition, i.e. the determination of its inverse or semi-inverse.

that the predictors (4.4a) or (4.4b) can be used along the whole length of the path in all practical cases. Successful application of these estimates is based on monotonically decreasing values of $D(s_i)$ or $\omega_1(s_i)$ and in general this condition is not fulfilled in the complete interval $0 \leqq s \leqq s^*$. Examples of cases where D first increases for $0 \leqq s \leqq s_1$ are given in [28, 31]. Consequently, implementation of the estimates (4.4a) and (4.4b) must be accompanied with appropriate safeguards.

(b) It sometimes happens that the path contains a stationary point $\dot{\rho} = 0$ or a point where $\dot{\rho} \cong 0$, after which ρ starts to increase again [31]. The advantage of the parameter s as compared to ρ is clearly demonstrated in such cases.

(c) Just after the completion of this manuscript, the author received a study by Menzel and Schwetlick [32], which concerns a method quite similar to the one denoted here by basic procedure. It is shown in a rigorous fashion that the method converges if the solution curve $\bar{x}(s)$ exists, the matrix $[f_{i,\alpha}(x)]$ has the rank N and the stepsize Δs is taken small enough. These conditions correspond to the conditions stated in Section 3.4.

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APPENDIX A

The equivalence of conditions (2.26a) and (2.31b) folows from the connection between the properties of the matrices $[f_{i,j}]$ and $[A_{i,j}]$. The characteristic values and vectors $\{\omega(i), \bar{a}(i)\}$ of the symmetric matrix $[f_{i,j}]$ are defined by

$$F_{i,j}a_j - \omega\delta_{ij}a_j = 0. \tag{A1}$$

At the critical state

$$0 = \omega(1) < \omega(2) \leqslant \omega(3) \leqslant \omega(4) \leqslant \cdots\cdots \leqslant \omega(N) \tag{A2}$$

according to the definition given in (2.18). The symmetric matrix $[A_{i,j}]$ is defined by (2.29)

$$A_{i,j} = \frac{\partial D}{\partial f_{j,i}}. \tag{A3}$$

It satisfies the property

$$A_{i,j}f_{j,k} = f_{i,j}A_{j,k} = \delta_{ik}D. \tag{A4}$$

The characteristic values and vectors $\{\lambda(i)$: $\bar{b}(i)\}$ are defined by

$$A_{i,j}b_j - \lambda\delta_{ij}b_j = 0. \tag{A5}$$

They can be expressed in terms of $\omega(i)$ and $\bar{a}(i)$

$$\lambda(i) = \frac{D}{\omega(i)};\ \bar{b}(i) = \bar{a}(i) \tag{A6}$$
$$(i = 1, 2, 3, \ldots\ldots N).$$

At the critical state the eigenvalues $\lambda(i)$ become

$$\lambda(1) = \prod_{k=2}^{N} \omega(k);\ \lambda(i) = 0 \tag{A7}$$
$$(i = 2, 3, \ldots\ldots N).$$

The characteristic vectors $\{\bar{a}(i)$: $i = 1, 2, \ldots, N\}$ form a basis of a N-dimensional linear space R_N. The set of vectors $\bar{y}$ obtained by the operation

$$y_i = f_{i,j}x_j \tag{A8}$$

for all $\bar{x}$ that belong to R_N, defines a subspace $T(f_{i,j})$, called the range of $[f_{i,j}]$. In the particular case considered here this space is $(N-1)$ dimensional. A basis of it is: $\{\bar{a}(k)$; $k = 2,3, \ldots N\}$. If $\bar{x}$ belongs to the subspace $N(f_{i,j})$ with basis $\bar{a}(1)$, the operation (A8) yields $\bar{y} = 0$. The subspace $N(f_{i,j})$ is called the null space of $[f_{i,j}]$. It is noted that R_N is the union of $T(f_{i,j})$ and $N(f_{i,j})$.

The operation

$$y_i = A_{i,j}x_j \tag{A9}$$

for all $\bar{x}$ in R_N defines the range of $[A_{i,j}]$: $T(A_{i,j})$. This subspace has the basis $\{\bar{a}(1)\}$ and is equal to the null space of $[f_{i,j}]$. thus: $T(A_{i,j}) \equiv N(f_{i,j})$. Moreover, the set of all $\bar{x}$ leading to $\bar{y} = 0$ in (A9) defines the null space of $[A_{i,j}]$. A basis of this space is $\{\bar{a}(h)$: $h = 2, 3, 4, \ldots N\}$ and it is concluded that $N(A_{i,j}) \equiv T(f_{i,j})$. Apparently, the geometrical properties of $[f_{i,j}]$ and $[A_{i,j}]$ are complementary at the critical points.

Consider now the bifurcation condition (2.26b).

$$a_i(1)f_{i,0} = 0. \tag{A10}$$

This equation conveys that the vector $[f_{i,0}] \neq \bar{0}$ does not belong to $N(f_{i,j})$ but to $T(f_{i,j}) \equiv N(A_{i,j})$. Consequently, if (A10) is satisfied

$$A_{i,j}f_{j,0} = \frac{\partial D}{\partial f_{j,i}}f_{i,0} = 0 \tag{A11}$$

must follow. The argument can also be reversed. Consequently, (A10) and (A11) are equivalent statements.